

Critical Cosmology in Higher Order Gravity

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We construct the higher order terms of curvatures in Lagrangians of the scale factor for the Friedmann-Lemaître-Robertson-Walker universe, which are linear in the second derivative of the scale factor with respect to cosmic time. It is shown that they are composed from the Lovelock tensors at the first step; iterative construction yields arbitrarily high order terms. The relation to the former work on higher order gravity is discussed. Despite the absence of scalar degrees of freedom in cosmological models which come from our Lagrangian, it is shown that an inflationary behavior of the scale factor can be found. The application to the thick brane solutions is also studied.

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I. INTRODUCTION

It is well known that higher-derivative gravity has a scalar degree of freedom in general [1–3]. In cosmological models of higher-derivative gravity, the scalar mode is expected to play an important role [4–6]. On the other hand, some cases are also known that higher order terms in curvatures for a gravitational action do not affect cosmological development of a scale factor. For example, it is known that terms which consist of contraction of Weyl tensors in a gravitational Lagrangian do not change evolutionary equations for a scale factor in a model with homogeneous and isotropic space. The other special combinations of curvatures are known. In the specific dimension, the Euler form as a Lagrangian does not produce the dynamics of gravity at all, because the action becomes a topological quantity in such a case.

The dimensionally continued Euler densities have also been studied [7–13], because of their relation to the effective Lagrangian of string theory, and are found to give no scalar mode since the second derivative of the metric disappears in the action if we perform integration by parts. The absence of scalar modes is interesting for studying black holes in the theory, because the scalar modes lead to singularities, in general, which avoid expected horizons.

In recent years, it turned out that there is a special case where a scalar mode disappears in higher-derivative gravity. Originally, this fact was found in research of a three-dimensional theory of massive gravity [14] and an extended version in four dimensions was proposed [15]. The authors of those papers intended to study the renormalizability and unitarity of gravitation theory in a maximally-symmetric spacetime. Thus, the absence of a massive scalar mode is at least a necessary condition of such theories referred as critical gravity.¹ Until now, however, only the cases with curvature tensors of a limited number have been investigated in higher dimensions [16, 17]. We are interested in the higher order theory of gravitation in which a scalar mode does not appear in a general higher dimensions.

In the present paper, we generalize the structure of the Lagrangian of critical gravity, to models with higher order terms in curvature tensors in higher dimensions. We show that such extensions can be attained by use of the Lovelock tensors. In order to offer a systematic way to construct the required higher order term, we take an explanatory approach by assuming the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. In this approach, the absence of

¹ In our analysis, we do not care for the other critical value for the cosmological constant, etc.

second derivatives of the scale factor from the Lagrangian with appropriate total derivatives is considered as a necessary condition for disappearing scalar modes.

It should be noted that the combination defined in D -dimensional spacetime

$$R_{\mu\nu}R^{\mu\nu} - \frac{D}{4(D-1)}R^2, \quad (1.1)$$

is used in critical gravities in three and four dimensions [14, 15]. This term can be considered as a trace of multiplication of the Einstein tensor and a linear combination of the Einstein tensor and its trace part:

$$\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right) \left[R^{\mu\nu} - \frac{1}{2(D-1)}R g^{\mu\nu}\right] = G_{\mu\nu} \left(G^{\mu\nu} - \frac{1}{D-1}G^\lambda_\lambda g^{\mu\nu}\right), \quad (1.2)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ is the Einstein tensor. Incidentally, $R^{\mu\nu} - \frac{1}{2(D-1)}R g^{\mu\nu}$ is known as the Schouten tensor up to the factor $(D-2)^{-1}$. Now, if the FLRW metric is assumed, the time-time component of the Einstein tensor does not include the second time-derivative of the scale factor. The Schouten tensor appearing the above term is made for its spatial component to have no second time-derivative of the scale factor. Therefore, the trace of the product of two tensors is linear in the second time-derivative of the scale factor, and if a surface term is suitably assigned, the Lagrangian is expressed only with the scale factor and its first time-derivative. Thus, additional scalar modes do not appear. From this observation, we find that it can be extended by using the Lovelock tensor instead of the Einstein tensor when the dimension of spacetime is higher. In the present paper, we do not analyze the massive tensor modes in our models. Thus, the genuine criticality as quantum gravity is left for future works.

The FLRW geometry is known to be conformally flat [18], i.e., the Weyl tensor for the FLRW cosmological metrics vanishes. An extension of Lovelock gravity for conformally-flat geometry was considered by Meissner and Olechowski [19]. They showed that the extension is possible for a $(R)^n$ term in D dimensions provided $n < D$, whereas the Lovelock gravity has the $(R)^n$ term at most $2n < D$ (where (R) denotes a general curvature tensor). Oliva and Ray also construct higher-derivative gravity with the second order equation utilizing the Weyl and Riemann tensors [20]. Their Lagrangian involves up to $(R)^n$ term with $2n < D$ because of the use of the generalized Kronecker delta as in the case of the Lovelock gravity.

In the present paper, we show that it is possible to continue the higher-curvature terms beyond the number of dimensions.

The outline of this paper is as follows. In §2, we construct the candidate Lagrangian for cosmological models without scalar modes in the tensorial form as the first step. The confirmation of the property of the model Lagrangian is performed in §3, substituting the FLRW metric. In §4, we show that an extension to more higher order terms in curvatures can be obtained. Using the higher order Lagrangian so far obtained, we propose (toy) models for the scale factor with inflationary behavior is shown in §5. In §6, the application of our Lagrangian to constructing domain wall solutions is studied. In the last section, we offer some concluding remarks.

II. LOVELOCK TENSORS AND GENERALIZATION OF THE HIGHER ORDER TERM IN CRITICAL GRAVITY

In this section, we construct the higher order term in curvatures by generalizing that of critical gravity. We will verify the absence of scalar modes in cosmological models with the terms in the next section. First, we introduce the dimensionally continued Euler density

$$L^{(n)} = 2^{-n} \delta_{\lambda_1 \rho_1 \dots \lambda_n \rho_n}^{\sigma_1 \tau_1 \dots \sigma_n \tau_n} R^{\lambda_1 \rho_1}_{\sigma_1 \tau_1} \dots R^{\lambda_n \rho_n}_{\sigma_n \tau_n} , \quad (2.1)$$

where the generalized Kronecker delta is defined as

$$\delta_{\nu_1 \nu_2 \dots \nu_p}^{\mu_1 \mu_2 \dots \mu_p} = \begin{vmatrix} \delta_{\nu_1}^{\mu_1} & \delta_{\nu_2}^{\mu_1} & \dots & \delta_{\nu_p}^{\mu_1} \\ \delta_{\nu_1}^{\mu_2} & \delta_{\nu_2}^{\mu_2} & \dots & \delta_{\nu_p}^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\nu_1}^{\mu_p} & \delta_{\nu_2}^{\mu_p} & \dots & \delta_{\nu_p}^{\mu_p} \end{vmatrix} . \quad (2.2)$$

The dimensionally continued Euler density $L^{(n)}$ consists of n -th order in the curvature tensors $((R)^n)$. For example, for $n = 1$, we find the Einstein-Hilbert term

$$L^{(1)} = R , \quad (2.3)$$

and for $n = 2$, we find the Gauss-Bonnet term

$$L^{(2)} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 , \quad (2.4)$$

as is well known.

Next, we consider the Lovelock tensor [7]. The Lovelock tensor is a generalization of the Einstein tensor, and defined as,

$$G^{(n)\mu}_{\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta(\int d^D x \sqrt{-g} L^{(n)})}{\delta g^{\rho\nu}} g^{\mu\rho} = -2^{-(n+1)} \delta^{\mu\sigma_1\tau_1\cdots\sigma_n\tau_n}_{\nu\lambda_1\rho_1\cdots\lambda_n\rho_n} R^{\lambda_1\rho_1}_{\sigma_1\tau_1} \cdots R^{\lambda_n\rho_n}_{\sigma_n\tau_n}. \quad (2.5)$$

This is a symmetric tensor of n -th order in the curvature tensors. For example, for $n = 1$, we find

$$G^{(1)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \quad (2.6)$$

and this is known as the Einstein tensor especially. For $n = 2$, we find

$$G^{(2)}_{\mu\nu} = 2(R_{\mu\rho\sigma\tau} R_{\nu}^{\rho\sigma\tau} - 2R_{\mu\rho\nu\sigma} R^{\rho\sigma} - 2R_{\mu\rho} R_{\nu}^{\rho} + R R_{\mu\nu}) - \frac{1}{2} L^{(2)} g_{\mu\nu}. \quad (2.7)$$

It should be noted that

$$G^{(n)} \equiv G^{(n)\lambda}_{\lambda} = \frac{2n - D}{2} L^{(n)}, \quad (2.8)$$

where D denotes the dimension of spacetime.

Here, we construct the new combination of the Lovelock tensor and the metric multiplied by the trace of the Lovelock tensor. That is,

$$S^{(n)\mu\nu} \equiv G^{(n)\mu\nu} - \frac{1}{D-1} G^{(n)} g^{\mu\nu}. \quad (2.9)$$

It is worth noting that

$$G^{(n)}_{\mu\nu} = (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\nu} g_{\rho\sigma}) S^{(n)\rho\sigma}. \quad (2.10)$$

For example, for $n = 1$, we obtain

$$S^{(1)\mu\nu} = R^{\mu\nu} - \frac{1}{2(D-1)} R g^{\mu\nu}, \quad (2.11)$$

which is proportional to the Schouten tensor. Therefore we obtain the following combination

$$G^{(1)}_{\mu\nu} S^{(1)\mu\nu} = R_{\mu\nu} R^{\mu\nu} - \frac{D}{4(D-1)} R^2, \quad (2.12)$$

which appears in critical gravities [14, 15].

Now, we find that the natural generalization of this is given by

$$G^{(n)}_{\mu\nu} S^{(n')\mu\nu} = G^{(n)}_{\mu\nu} G^{(n')\mu\nu} - \frac{1}{D-1} G^{(n)} G^{(n')}. \quad (2.13)$$

It is worth pointing out that the expression is symmetric against the exchange of n and n' .

In the next section, we confirm that this combination is suitable for an extension of critical gravity in higher dimensions, by utilizing the FLRW metric.

III. HIGHER ORDER TERM FOR FLRW METRIC

We consider the following FLRW metric in D dimensions:

$$ds^2 = -dt^2 + a(t)^2 d\Omega_{D-1}^2, \quad (3.1)$$

where $a(t)$ is the scale factor and $d\Omega_{D-1}^2$ denotes the line element of a maximally symmetric space of $(D-1)$ -dimensions, whose scalar curvature is normalized to $(D-1)(D-2)k$ with $k = 1, 0, -1$.

First in this section, we examine the Lovelock tensors of $(R)^n$. By explicit calculation of curvatures, we find, for $n = 1$,

$$G^{(1)0}_0 = -\frac{(D-1)(D-2)}{2} \frac{\dot{a}^2 + k}{a^2}, \quad (3.2)$$

$$G^{(1)i}_j = -\left[(D-2)\frac{\ddot{a}}{a} + \frac{(D-2)(D-3)}{2} \frac{\dot{a}^2 + k}{a^2}\right] \delta^i_j, \quad (3.3)$$

where $\dot{a} = \frac{da}{dt}$ and $\ddot{a} = \frac{d^2a}{dt^2}$. Here and hereafter, we use the suffixes denoting the spatial dimensions, $i, j = 1, 2, \dots, D-1$. Also, for $n = 2$, we obtain

$$G^{(2)0}_0 = -\frac{(D-1)(D-2)(D-3)(D-4)}{2} \left(\frac{\dot{a}^2 + k}{a^2}\right)^2, \quad (3.4)$$

$$G^{(2)i}_j = -\left[2(D-2)(D-3)(D-4)\frac{\ddot{a}}{a} \frac{\dot{a}^2 + k}{a^2} + \frac{(D-2)(D-3)(D-4)(D-5)}{2} \left(\frac{\dot{a}^2 + k}{a^2}\right)^2\right] \delta^i_j. \quad (3.5)$$

By the combinatorial property of the generalized Kronecker delta, we can find the Lovelock tensor for a general n as follows:

$$G^{(n)0}_0 = -\frac{(D-1)(D-2)\cdots(D-2n)}{2} \left(\frac{\dot{a}^2 + k}{a^2}\right)^n, \quad (3.6)$$

$$G^{(n)i}_j = -\left[n(D-2)(D-3)\cdots(D-2n)\frac{\ddot{a}}{a} \left(\frac{\dot{a}^2 + k}{a^2}\right)^{n-1} + \frac{(D-2)(D-3)\cdots(D-2n-1)}{2} \left(\frac{\dot{a}^2 + k}{a^2}\right)^n\right] \delta^i_j, \quad (3.7)$$

and $G^{(n)0}_j = G^{(n)i}_0 = 0$. It should be noted that the 00 component of the Lovelock tensor does not include \ddot{a} .

Next, we calculate the generalized Schouten tensor $S_{(n)\nu}^\mu$ for the FLRW metric. Because the trace of the Lovelock tensor is given as

$$\frac{1}{D-1}G^{(n)} = - \left[n(D-2)(D-3)\cdots(D-2n)\frac{\ddot{a}}{a}\left(\frac{\dot{a}^2+k}{a^2}\right)^{n-1} + (D-2n)\frac{(D-2)(D-3)\cdots(D-2n)}{2}\left(\frac{\dot{a}^2+k}{a^2}\right)^n \right], \quad (3.8)$$

we find

$$S_{(n)0}^0 = n(D-2)(D-3)\cdots(D-2n)\frac{\ddot{a}}{a}\left(\frac{\dot{a}^2+k}{a^2}\right)^{n-1} - (2n-1)\frac{(D-2)(D-3)\cdots(D-2n)}{2}\left(\frac{\dot{a}^2+k}{a^2}\right)^n, \quad (3.9)$$

$$S_{(n)j}^i = \frac{(D-2)(D-3)\cdots(D-2n)}{2}\left(\frac{\dot{a}^2+k}{a^2}\right)^n \delta_j^i. \quad (3.10)$$

By construction, \ddot{a} is absent in $S_{(n)j}^i$.

Now, we consider the combined term $G^{(n)}_{\mu\nu}S_{(n')\mu\nu}^{\mu\nu}$. We obtain

$$G^{(n)}_{\mu\nu}S_{(n')\mu\nu}^{\mu\nu} = -\frac{1}{4}(D-1)[(D-2)(D-3)\cdots(D-2n)][(D-2)(D-3)\cdots(D-2n')] \times \left[2(n+n')\frac{\ddot{a}}{a}\left(\frac{\dot{a}^2+k}{a^2}\right)^{n+n'-1} + (D-2(n+n'))\left(\frac{\dot{a}^2+k}{a^2}\right)^{n+n'} \right]. \quad (3.11)$$

Because this combination is apparently linear in \ddot{a} , the action including this term can be expressed as the functional of a and \dot{a} , by means of integrations by part. Therefore, we realize that there is no scalar mode in a cosmological setting.

The combination $L^{(n)(n')} \equiv -4G^{(n)}_{\mu\nu}S_{(n')\mu\nu}^{\mu\nu}$ can exist for limited numbers (n, n') , which depends on D . For the exchange symmetry, we assume $n \leq n'$. We find that $L^{(n)(n')}$ is non-zero when $(n, n') = (1, 1)$ for $D = 3, 4$, $(n, n') = (1, 1), (1, 2), (2, 2)$ for $D = 5, 6$ and $(n, n') = (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)$ for $D = 7, 8$, etc. Therefore, we have new higher-derivative terms including $(R)^m$ with $m \leq D-1$ for odd D and $m \leq D-2$ for even D .

Up to now, we become aware of a similarity to the Lovelock Lagrangian. It takes the form for the FLRW metric as follows:

$$L^{(m)} = (D-1)(D-2)\cdots(D-2m+1) \times \left[2m\frac{\ddot{a}}{a}\left(\frac{\dot{a}^2+k}{a^2}\right)^{m-1} + (D-2m)\left(\frac{\dot{a}^2+k}{a^2}\right)^m \right]. \quad (3.12)$$

The equivalence up to the overall constant is obvious, that is $L^{(n)(n')} \propto L^{(n+n')}$. Incidentally, we can consider $L^{(0)(n)}$ as the Lovelock Lagrangian $L^{(n)}$. We note, however, that the Lovelock Lagrangian $L^{(m)}$ has its meaning as a part of Lagrangian when $m = 1$ for $D = 3, 4$, $m = 1, 2$ for $D = 5, 6$ and $m = 1, 2, 3$ for $D = 7, 8$, etc., because $L^{(m)}$ becomes a total derivative for $D = 2m$.

Before closing this section, discussion on relation to the work of Meissner and Olechowski [19] is in order. Their approach is equivalent to considering the Lagrangian constructed from the Lovelock Lagrangian in which the Riemann tensor is expressed by the Schouten tensor under the assumption of vanishing Weyl tensor.² They used the combinatorial property of the generalized Kronecker delta for extension to higher order in curvatures. If the FLRW metric is substituted, we find that the Lagrangian of Meissner and Olechowski of order of $(R)^m$ coincides with our Lagrangian $L^{(n)(n')}$, where $n + n' = m$, up to the constant factor. The allowed spacetime dimension is the same for odd m , $m < D$. Therefore, our Lagrangian and theirs are almost equivalent. The variety with respect to two integers in $L^{(n)(n')}$ is due to the use of Riemann tensors as well as scalar curvatures and Ricci tensors in our approach. It is notable that differences may occur if we consider the black hole or non-conformally flat solutions in the theory governed by the Lagrangians of higher order terms.

Later, we show that the restriction by the dimensions can be overcome. To exhibit the discussion on the subject, we examine the cosmological action in the present model again in the next section.

IV. FURTHER EXTENSION TO HIGHER ORDER IN CURVATURE (ESPECIALLY FOR $m > D$)

We consider the action for $m \geq 2$:

$$S^{(m)} = \int d^D x \sqrt{-g} \sum_{n=0}^m \alpha_{(n)(m-n)} L^{(n)(m-n)}, \quad (4.1)$$

with arbitrary coefficients $\alpha_{(n)(m-n)}$. Here, we regard $L^{(0)(n)}$ as $L^{(n)}$.

Then, the action for the scale factor $a(t)$ can be read as

$$S^{(m)} \propto \beta_m \int dt a^{D-1} \left[2m \frac{\ddot{a}}{a} \left(\frac{\dot{a}^2 + k}{a^2} \right)^{m-1} + (D - 2m) \left(\frac{\dot{a}^2 + k}{a^2} \right)^m \right], \quad (4.2)$$

² Strictly speaking, they replace Riemann tensor by the Kulkarni-Nomizu product of the Schouten tensor with the metric in the Lovelock tensor.

where

$$\beta_m = (D-1)\alpha_{(n)(m-n)} \frac{(D-2)!}{(D-2n-1)!} \frac{(D-2)!}{(D-2(m-n)-1)!}. \quad (4.3)$$

Using the expansion

$$(\dot{a}^2 + k)^{m-1} \ddot{a} = \sum_{\ell=0}^{m-1} \binom{m-1}{\ell} k^{m-1-\ell} \dot{a}^{2\ell} \ddot{a} = \sum_{\ell=0}^{m-1} \frac{k^{m-1-\ell}}{2\ell+1} \binom{m-1}{\ell} \frac{d}{dt} \dot{a}^{2\ell+1}, \quad (4.4)$$

we rewrite the action, after partial integration, as

$$I^{(m)}[a, \dot{a}] \equiv (D-2m)\beta_m \int dt a^{D-1} \left[-2m \sum_{\ell=0}^{m-1} \frac{k^{m-1-\ell}}{2\ell+1} \binom{m-1}{\ell} \frac{\dot{a}^{2\ell+2}}{a^{2m}} + \left(\frac{\dot{a}^2 + k}{a^2} \right)^m \right] \quad (4.5)$$

Especially for $k=0$, we find a simple action

$$I^{(m)} = (D-2m)\beta_m \int dt a^{D-1} \left[-\frac{1}{2m-1} \left(\frac{\dot{a}^2}{a^2} \right)^m \right]. \quad (4.6)$$

Now, we define the Lagrangian $L^{(m)}(a, \dot{a})$ for the scale factor $a(t)$ by

$$I^{(m)} = \int dt L^{(m)}(a, \dot{a}). \quad (4.7)$$

Then, we find the equation of motion for the scale factor

$$\begin{aligned} & \frac{d}{dt} \frac{\partial L^{(m)}}{\partial \dot{a}} - \frac{\partial L^{(m)}}{\partial a} \\ &= -(D-2m)\beta_m a^{D-2} \left[2m \frac{\ddot{a}}{a} \left(\frac{\dot{a}^2 + k}{a^2} \right)^{m-1} + (D-2m-1) \left(\frac{\dot{a}^2 + k}{a^2} \right)^m \right]. \end{aligned} \quad (4.8)$$

The Hamiltonian constraint is regarded as the variation equation $\frac{\partial L}{\partial N} = 0$, where N is the lapse function defined as $dt = N(t')dt'$. We set $N=1$ after the manipulation. We now find the Hamiltonian constraint equation

$$\left. \frac{\partial L^{(m)}}{\partial N} \right|_{N=1} = (D-2m)\beta_m a^{D-1} \left(\frac{\dot{a}^2 + k}{a^2} \right)^m. \quad (4.9)$$

It is known that the variation of the lapse function corresponds to the variation of g_{00} and the variation of the scale factor corresponds to the variation of g_{ii} , up to certain factors. We can find, from (4.9) and (4.8), the following generalized Lovelock tensor as the variation of the action with respect to the metric:

$$\bar{G}^{(m)0}_0 = -\frac{(D-2m)\beta_m}{2} \left(\frac{\dot{a}^2 + k}{a^2} \right)^m, \quad (4.10)$$

$$\bar{G}^{(m)i}_j = -\frac{(D-2m)\beta_m}{2(D-1)} \left[2m \frac{\ddot{a}}{a} \left(\frac{\dot{a}^2 + k}{a^2} \right)^{m-1} + (D-2m-1) \left(\frac{\dot{a}^2 + k}{a^2} \right)^m \right] \delta^i_j. \quad (4.11)$$

The generalized Lovelock tensor has a tensorial form, which is proportional to

$$\bar{G}_{\mu\nu}^{(m)} \propto \frac{1}{\sqrt{-g}} \frac{\delta S^{(m)}}{\delta g^{\mu\nu}}, \quad (4.12)$$

as seen from the construction, but with arbitrary coefficients in the definition of the action $S^{(m)}$. In spite of the arbitrariness, the functional form of the Lagrangian is unambiguous if the conformally flat metric is substituted.

In the similar manner, the corresponding generalized Schouten tensor is defined as in the previous section,

$$\bar{S}_{(m)}^{\mu\nu} \equiv \bar{G}^{(m)\mu\nu} - \frac{1}{D-1} \bar{G}^{(m)} g^{\mu\nu}, \quad (4.13)$$

with $\bar{G}^{(m)} \equiv g_{\mu\nu} \bar{G}^{(m)\mu\nu}$. Then we find

$$\bar{S}_{(m)0}^0 = \frac{(D-2m)\beta_m}{2(D-1)} \left[2m \frac{\ddot{a}}{a} \left(\frac{\dot{a}^2 + k}{a^2} \right)^{m-1} - (2m-1) \left(\frac{\dot{a}^2 + k}{a^2} \right)^m \right], \quad (4.14)$$

$$\bar{S}_{(m)j}^i = \frac{(D-2m)\beta_m}{2(D-1)} \left(\frac{\dot{a}^2 + k}{a^2} \right)^m \delta_j^i. \quad (4.15)$$

Then, the trace of the product of these tensors is found to be

$$\begin{aligned} L^{(m)(m')} &= -4 \bar{G}_{\mu\nu}^{(m)} \bar{S}_{(m')}^{\mu\nu} \\ &= \frac{(D-2m)(D-2m')\beta_m\beta_{m'}}{D-1} \left[2(m+m') \frac{\ddot{a}}{a} + \{D-2(m+m')\} \frac{\dot{a}^2 + k}{a^2} \right] \left(\frac{\dot{a}^2 + k}{a^2} \right)^{m+m'-1} \end{aligned} \quad (4.16)$$

Again, we obtain the same functional form as those constructed from the Lovelock tensors in the previous section. More iterative operations yield the more higher order terms. For this time, however, no limitation on the relation between the number of dimensions D and the order of curvatures $\ell \equiv m + m'$, except for the case with $2\ell = D$ (the Lagrangian becomes a total derivative in this case for the conformally flat spacetime). Although we do not exhibit an explicit tensorial form of the higher order term in mass dimensions, obviously it can be expressed as the combination of curvature tensors and their covariant derivatives.

Unfortunately, only for $D = 4$, our iterative method cannot give a $(R)^3$ term, since the $(R)^2$ term becomes a total derivative for the conformally flat metric. We define the expression for $L^{(3)}$ by the manner followed by Meissner and Olechowski as

$$L^{(3)} \propto \delta_{\nu_1\nu_2\nu_3}^{\mu_1\mu_2\mu_3} S_{(1)\mu_1}^{\nu_1} S_{(1)\mu_2}^{\nu_2} S_{(1)\mu_3}^{\nu_3}. \quad (4.17)$$

Incidentally, it is obvious that $L^{(3)}$ is at most linear in \ddot{a} if the FLRW metric is substituted, because only $S_{(1)0}^0$ is the component of the Schouten tensor which is linear in \ddot{a} . Then, the

desired terms of all the order in curvatures and derivatives in any dimensions can be created by our iterative method.

Another way to cross the dimensional limitation, which is inspired by the work of Meissner and Olechowski [19], is also found. In the manner of their paper [19], the following Lagrangian (modulo volume form) is proposed:

$$\delta_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} S_{(1)\mu_1}^{\nu_1} \dots S_{(1)\mu_n}^{\nu_n}, \quad (4.18)$$

where it is expressed by our present notation. We become aware of an extension of it as

$$\delta_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} \bar{S}_{(m_1)\mu_1}^{\nu_1} \dots \bar{S}_{(m_n)\mu_n}^{\nu_n}, \quad (4.19)$$

where m_i ($i = 1, \dots, n$) are arbitrary integers. For the FLRW metric, this term turns out to be proportional to

$$2\ell \frac{\ddot{a}}{a} \left(\frac{\dot{a}^2 + k}{a^2} \right)^{\ell-1} + (D - 2\ell) \left(\frac{\dot{a}^2 + k}{a^2} \right)^{\ell}, \quad (4.20)$$

where $\ell = \sum_{i=1}^n m_i$. This is also the same form of the candidate Lagrangian. Many equivalent combinations exist for higher order terms under the vanishing-Weyl-tensor condition.

In the next section and after, we will turn our attention to apply the higher order Lagrangian obtained here to cosmological models .

V. $f(H^2)$ COSMOLOGY

In this section, we investigate the possible inflationary stage in evolution of the universe in the model with higher order terms discussed in the previous sections.

Let us consider the general cosmological action

$$L = a^{D-1} \sum_{\ell=0}^{\infty} \beta_{\ell} \left[2\ell \frac{\ddot{a}}{a} \left(\frac{\dot{a}^2 + k}{a^2} \right)^{\ell-1} + (D - 2\ell) \left(\frac{\dot{a}^2 + k}{a^2} \right)^{\ell} \right], \quad (5.1)$$

with an appropriate total derivative term with respect to the time, which removes the second derivative of the scale factor with respect to time.

As usual, the energy momentum tensor of matter is taken as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}, \quad (5.2)$$

with the action for matter S_{matter} . The non-zero components of the energy momentum tensor of matter is assumed as $T_0^0 = -\rho$ and $T_j^i = p\delta_j^i$, where ρ is the energy density and p is the pressure, for the FLRW universe. As is seen in the previous sections, the equation of motion derived from the action (5.1) coincides with the linear combination of the components of the Lovelock tensors in functional form with the assumption of the FLRW metric. Thus, the energy conservation

$$\dot{\rho} + (D-1)\frac{\dot{a}}{a}(\rho + p) = 0, \quad (5.3)$$

holds generally. This fact is due to the absence of the scalar mode and the fact that we did not need to rescale the metric.

We here give a few model in four-dimensional spacetime with focus on the possibility of an inflationary growth of the scale factor. Furthermore, we take $k = 0$, i.e., assume the flat space. Then, the action (5.1) is equivalent to

$$L(a, \dot{a}) = a^3 \sum_{\ell=0}^{\infty} \beta_{\ell} \left[-\frac{4-2\ell}{2\ell-1} \left(\frac{\dot{a}^2}{a^2} \right)^{\ell} \right]. \quad (5.4)$$

The 00 component of the equation of motion reads

$$\sum_{\ell=0}^{\infty} \beta_{\ell} \frac{4-2\ell}{2} \left(\frac{\dot{a}^2}{a^2} \right)^{\ell} = \rho. \quad (5.5)$$

If we can choose the coefficient β_{ℓ} freely, almost arbitrary equations including ρ and $H \equiv \frac{\dot{a}}{a}$ can be made. That is,

$$f(H^2) = \frac{8\pi G}{3}\rho, \quad (5.6)$$

where $f(x)$ is a function which can be expressed by a series and does not include the x^2 term. Here the factor given in the right hand side is chosen as for the similarity with the standard cosmology. We wish to call the cosmology of the model ‘ $f(H^2)$ cosmology’.³

Now, we specify the function $f(x)$. For example, let us consider

$$H^2 \left(1 - \frac{H^4}{M^4} \right)^{-1} = \frac{8\pi G}{3}\rho, \quad (5.7)$$

with a typical mass scale M . Solving the equation for H^2 , we obtain

$$H^2 = \frac{8\pi G}{3}\rho \left[\frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{8\pi G\rho}{3M^2} \right)^2} \right]^{-1}. \quad (5.8)$$

³ If $k \neq 0$, we obtain ‘ $f(H^2 + k/a^2)$ cosmology’.

If the energy density is sufficiently low, such as $\rho \ll G^{-1}M^2$, the relation in the standard cosmology is valid:

$$H^2 \approx \frac{8\pi G}{3}\rho. \quad (5.9)$$

On the other hand, in the era of the high energy density such that $\rho \gg G^{-1}M^2$, we find

$$H^2 \approx M^2, \quad (5.10)$$

and we obtain an approximate de Sitter inflation.

Another model can be chosen, in which the correction is a monomial of the higher order term. That is,

$$H^2 + M^2 \left(\frac{H^2}{M^2} \right)^m = \frac{8\pi G}{3}\rho \quad (m \geq 3). \quad (5.11)$$

In the rapid expansion phase, $H^2 \gg M^2$, the correction would be dominant. Further, if we assume the dust matter, i.e., $\rho \propto a^{-3}$, we obtain

$$a(t) \propto t^{\frac{2m}{3}}. \quad (5.12)$$

This indicates the phase of the power-law inflation.

Even though we show special toy models here, we find that the existence of higher order terms yields the inflationary growth of the scale factor with ordinary matter, and with no scalar mode and no redefinition of the metric.

VI. DOMAIN WALL

We shall apply the previous discussion on a conformally flat metric to solutions for domain walls (“thick branes”). The D -dimensional metric suitable for a co-dimension one domain wall is given by

$$ds^2 = e^{2A(y)}(-dt^2 + d\mathbf{x}^2) + dy^2, \quad (6.1)$$

where $\mathbf{x} = (x^1, x^2, \dots, x^{D-2})$. In addition, we consider a neutral, minimally-coupled, self-interacting scalar field as a classical matter field. It is known that superpotential method [21–25] is available in this case to find BPS kink equations. Let us investigate the case with higher order terms in a similar manner.

Substituting the metric (6.1), the gravitational Lagrangian becomes

$$\mathcal{L} = e^{(D-1)A} \sum_{\ell=0}^{\infty} (-1)^\ell \beta_\ell \left[2\ell A'' (A')^{2\ell-2} + D (A')^{2\ell} \right], \quad (6.2)$$

where the prime (') denotes a derivative with respect to y . The term in A'' can be removed by discarding a total divergence. Thus we obtain

$$L(A, A') = e^{(D-1)A} \sum_{\ell=0}^{\infty} (-1)^\ell \beta_\ell \left[-\frac{D-2\ell}{2\ell-1} (A')^{2\ell} \right]. \quad (6.3)$$

The action for the real scalar field $\phi(y)$ for static branes is written by

$$L_S(A, \phi, \phi') = e^{(D-1)A} \left[-\frac{1}{2}(\phi')^2 - V(\phi) \right], \quad (6.4)$$

Since the coefficients β_ℓ can be arbitrarily chosen, we use an arbitrary function $F(x)$ to represent the general action. Then, we can write the total action as

$$L_{total}(A, A', \phi, \phi') = e^{(D-1)A} \left[F(A'^2) - \frac{1}{2}(\phi')^2 - V(\phi) \right]. \quad (6.5)$$

The field equations can now be derived in a usual manner. The differential equation for the scalar is

$$\phi'' + (D-1)A'\phi' - V_\phi = 0, \quad (6.6)$$

where $V_\phi = \frac{dV}{d\phi}$. The equation for A is written as

$$2[F_x(A'^2) + 2A'^2 F_{xx}(A'^2)]A'' - (D-1) \left[F(A'^2) - 2A'^2 F_x(A'^2) - \frac{1}{2}(\phi')^2 - V(\phi) \right] = 0, \quad (6.7)$$

where $F_x = \frac{dF(x)}{dx}$ and $F_{xx} = \frac{d^2 F(x)}{dx^2}$. The reparametrization invariance of y leads to the first integral given by

$$F(A'^2) - 2A'^2 F_x(A'^2) + \frac{1}{2}(\phi')^2 - V(\phi) = 0. \quad (6.8)$$

Define $h(x) \equiv 2xF_x(x) - F(x)$. Then, one can find $h_x(x) = \frac{dh}{dx} = F_x(x) + 2xF_{xx}(x)$. Note that $h(x)$ as well as $F(x)$ does not possess the $x^{D/2}$ term for even D . Using these, the equations can be rewritten as

$$2h_x(A'^2)A'' + (D-1)(\phi')^2 = 0, \quad (6.9)$$

and

$$-h(A'^2) + \frac{1}{2}(\phi')^2 - V(\phi) = 0. \quad (6.10)$$

Now, we take a BPS ansatz [21–26]

$$A' = -W(\phi). \quad (6.11)$$

Then, the equations reduces to the first order equation

$$\phi' = \frac{2}{D-1} h_x(W^2) W_\phi, \quad (6.12)$$

and the potential

$$V(\phi) = \frac{2}{(D-1)^2} [h_x(W^2)]^2 (W_\phi)^2 - h(W^2), \quad (6.13)$$

in terms of $W(\phi)$.

If the function $h(x)$ is a polynomial up to the quadratic order, a domain wall solution exists, for the potential $V(\phi)$ has two minima [21–26]. The solution can be expressed as a kink solution in the ϕ^4 theory. We find here that the potential with many vacua naturally corresponds to the general higher order gravity.

For example, we try to express the sine-Gordon equation. We take $W(\phi) = B\phi$, with a positive constant B . Further, we choose

$$h_x(x) = C \cos \frac{\alpha\sqrt{x}}{2}, \quad (6.14)$$

with constants α and C . This choice is possible if the spacetime dimension D is odd. Then the scalar obeys the sine-Gordon equation

$$\phi'' = -\frac{\alpha B^2 C}{D-1} \sin \frac{\alpha B \phi}{2} \phi' = -\frac{\alpha B^3 C^2}{(D-1)^2} \sin \alpha B \phi, \quad (6.15)$$

and an exact static solution is known as

$$\phi(y) = \frac{4}{\alpha B} \left(\arctan e^{\frac{\alpha B^2 C}{D-1} y} - \frac{\pi}{4} \right). \quad (6.16)$$

Then, the potential takes the form:

$$V(\phi) = \frac{2B^2 C^2}{(D-1)^2} \cos^2 \frac{\alpha B \phi}{2} - \frac{8C}{\alpha^2} \left(\frac{\alpha B \phi}{2} \sin \frac{\alpha B \phi}{2} + \cos \frac{\alpha B \phi}{2} \right) + V_0. \quad (6.17)$$

The minima of the potential are located at $\alpha B \phi / 2 = \pi / 2 \pm n\pi$ ($n = 0, 1, 2, \dots$). Although it is difficult to obtain exact solutions of other types, we can suppose multiple domain walls with distinct topological numbers in this and similar models. The model with many vacua may also serve an interesting mechanism to realize naturally a small cosmological constant in (thick) brane world. This possibility will be studied in future.

VII. SUMMARY AND CONCLUSION

In the present paper, we have attempted to show the possible quasi-linear second-order theory of gravity in conformally flat spacetimes. Models with arbitrary higher order of curvatures have been obtained. As long as we adopt an isotropic and homogeneous cosmological setting, the energy conservation holds in the models because there is no scalar mode and no requirement of frame rescaling. In spite of them, inflationary expansion can be found in the models. We have also found that the domain wall solution in the present type of the higher order gravity can be obtained naturally with the potential having many minima.

Our work corresponds to the extension of the Lovelock higher-curvature gravity in arbitrarily higher order terms in higher dimensions. Our analysis, however, has been limited for the case with the conformally flat spacetime and the coefficient on the tensorial form of the action has been still ambiguous. The stability of the solutions obtained here is problematic for anisotropic perturbations or tensor modes. To study it, we should classify the tensorial form of the Lagrangian which is equivalent only if the Weyl tensor vanishes. It is known that the dimensionally continued Euler forms have the property of factorization in terms of those of lower orders when the spacetime is represented by the direct product of spaces [10]. Thus, compactification should be worth studying to seek some special combination of curvature tensors. It is expected that the hopeful ‘critical’ relation among the coefficients of different orders of curvatures will be selected by consideration on various background spacetimes.

Nevertheless, we emphasize that our arbitrarily high order gravity can be applied to many models in various contexts. The problem of initial singularity can be reconsidered by studying classical bouncing universes in our model. On the other hand, the Wheeler-DeWitt equation of the Lagrangian should lead to higher-derivative quantum cosmology. The equation must be difficult to treat with, but the study on it may shed new light on quantum gravity. The possible black hole solutions are interesting in both cases of asymptotically flat and asymptotically AdS spacetime. We shall return to some of the problems in future.

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